Natural parameterizations of closed projective plane curves

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March 14, 2020

Abstract

A natural parametrization of smooth projective plane curves which tolerates the presence of sextactic points is the Forsyth-Laguerre parametrization. On a closed projective plane curve, which necessarily contains sextactic points, this parametrization is, however, in general not periodic. We show that by the introduction of an additional scalar parameter $\alpha \leq \frac{1}{2}$ one can define a projectively invariant 2π-periodic global parametrization on every simple closed convex sufficiently smooth projective plane curve without inflection points. For non-quadratic curves this parametrization, which we call balanced, is unique up to a shift of the parameter. The curve is an ellipse if and only if $\alpha = \frac{1}{2}$, and the value of α is a global projective invariant of the curve. The parametrization is equivariant with respect to duality.

Keywords: projective plane curve, Forsyth-Laguerre parametrization, global invariant
MSC 2010: 53A20, 52A10

1 Parameterizations of projective plane curves

Projective plane curves have been intensely studied in the second half of the 19-th and the beginning of the 20-th century and are a classical subject of differential geometry. In this paper we consider periodic parameterizations of closed projective plane curves. The well-known natural local parameterizations cannot in general be extended to the whole curve. We show that under some non-degeneracy assumptions there nevertheless exists a natural periodic global parametrization. On non-quadratic curves it gives rise to a projectively invariant metric on the curve.

The most natural way to represent curves in the real projective plane is by projective images of vector-valued solutions of third-order linear differential equations. This representation has already been studied in the 19-th century by Halphen, Forsyth, Laguerre, and others. For a detailed account see [10] or [1], for a more modern exposition see [8].

Let γ be a regularly parameterized (i.e., with non-vanishing tangent vector) curve of class C^k, $k \geq 5$, in $\mathbb{RP}^2$ without inflection points. Then there exist coefficient functions c_0, c_1, c_2 of class C^{k-3} such that γ is the projective image of a vector-valued solution $y(t)$ of the ODE

$$y'''(t) + c_2(t)y''(t) + c_1(t)y'(t) + c_0(t)y(t) = 0. \tag{1}$$

By multiplying the solution $y(t)$ by a non-vanishing scalar function we may achieve that the coefficient c_2 vanishes identically and that $\det(y'', y', y) \equiv 1$ [8, p. 30]. Subsequently decomposing the differential operator on the left-hand side of (1) in its skew-symmetric and symmetric part, we arrive at the ODE

$$[y'''(t) + 2\alpha(t)y'(t) + \alpha'(t)y(t)] + \beta(t)y(t) = 0 \tag{2}$$

with the coefficient functions

$$\alpha = \frac{1}{2}c_1 - \frac{1}{6}c_2^2 - \frac{1}{2}c_2', \quad \beta = c_0 - \frac{1}{3}c_1c_2 + \frac{2}{27}c_2^3 - \frac{1}{3}c_2'' - \alpha'$$

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being of class C^{k-4}, C^{k-5}, respectively [10, p. 16]. The lift y of γ is then of class C^{k-2}.

The function β transforms as the coefficient of a cubic differential $\beta(t)\,dt^3$ under reparametrizations of the curve γ. This differential is called the *cubic form* of the curve [8, pp. 15, 41]. Its cubic root $\sqrt[3]{\beta(t)}\,dt$ is called the *projective length element*, and its integral along the curve is the *projective arc length.* Points on γ where β vanishes are called *sextactic* points. In the absence of sextactic points the curve may hence be parameterized by its projective arc length, which is equivalent to achieving $\beta \equiv 1$ and is the most natural parametrization of a curve in the projective plane [1, p. 50].

A simple closed strictly convex curve has at least six sextactic points. This is the content of the six-vertex theorem [8, p. 73] which was first proven in [7], according to [9]. Therefore such a curve does not possess a global parametrization by projective arc length.

Another common way to parameterize curves in the projective plane is the *Forsyth-Laguerre* parametrization which is characterized by the condition $\alpha \equiv 0$ in (2). This parametrization is unique up to linear-fractional transformations of the parameter t [10, pp. 25–26], see also [1, pp. 48–50] and [8, p. 41]. This implies that the curve γ carries an invariant projective structure, which was called the *projective curvature* in [8, p. 15]. It is closely related to the projective curvature in the sense of [1, p. 107], which is defined as the value of the coefficient α in the projective arc length parametrization. Locally projective structures on closed curves in general have been studied in [5].

To (2) we may associate the second-order differential equation

$$x''(t) + \frac{1}{2}\alpha(t)x(t) = 0, \tag{3}$$

whose solution is of class C^{k-2}. It is not hard to check [4, p. 121] that if x_1, x_2 are linearly independent solutions of ODE (3), then the products x_1^2, x_1x_2, x_2^2 are linearly independent C^{k-2} solutions of the ODE

$$w'''(t) + 2\alpha(t)w'(t) + \alpha'(t)w(t) = 0 \tag{4}$$

which can be obtained from (2) by retaining the skew-symmetric part only. These solutions satisfy the homogeneous quadratic relation $x_1^2 \cdot x_2^2 = (x_1x_2)^2$. Hence the vector-valued solution of ODE (4) maps to the projective ellipse ς defined by this relation.

This construction is equivariant with respect to reparametrizations of the curve γ in the following sense [8, Theorem 1.4.3].

Lemma 1.1. *Let $t \mapsto s(t)$ be a reparametrization of the curve γ, and let $\tilde\alpha(s)$ be the corresponding coefficient in ODE* (2) *in the new parameter. Let $x(t)$ be a vector-valued solution of ODE* (3) *with linearly independent components. Then there exists a non-vanishing scalar function $\sigma(s)$ such that $\tilde x(s) = \sigma(s)x(t(s))$ is a vector-valued solution of the ODE $\frac{d^2\tilde x(s)}{ds^2} + \frac{1}{2}\tilde\alpha(s)\tilde x(s) = 0$.* □

Obviously the scalar $\sigma(s)$ may be chosen to be positive. In fact, if we restrict to reparametrizations satisfying $\frac{ds}{dt} > 0$ and normalize the solutions $x(t), \tilde x(s)$ such that $\det(x, \frac{dx}{dt}) = \det(\tilde x, \frac{d\tilde x}{ds}) \equiv 1$, then $\sigma(s) = \sqrt{\frac{ds}{dt}}$ [6, eq. (2)].

Now if two vector-valued functions $x(t), \tilde x(t)$ satisfying ODE (3) with coefficient functions $\alpha(t), \tilde\alpha(t)$, respectively, are related by a scalar factor, $\tilde x(t) = \sigma(t)x(t)$ for some non-vanishing σ, and $\det(x, x') = \det(\tilde x, \tilde x') \equiv 1$, then α and $\tilde\alpha$ coincide [8, Theorem 1.3.1]. We can then reformulate above lemma as follows.

Corollary 1.2. *Let $\gamma(t)$ be a curve in $\mathbb{RP}^2$ without inflection points, and let $y(t)$ be a lift of γ satisfying ODE* (2) *with some coefficient function $\alpha(t)$. Let $t \mapsto s(t)$ be a reparametrization of the curve γ. Let $x(t), \tilde x(s)$ be vector-valued solutions of ODE* (3) *with linearly independent components and with coefficient functions $\alpha(t), \tilde\alpha(s)$, respectively. Suppose further that $\det(x, \frac{dx}{dt}) = \det(\tilde x, \frac{d\tilde x}{ds}) \equiv 1$, and that there exists a non-vanishing scalar function $\sigma(s)$ such that $\tilde x(s) = \sigma(s)x(t(s))$ for all s. Then $\gamma(s)$ has a lift $\tilde y(s)$ which is a solution of ODE* (2) *with $\tilde\alpha(s)$ as the corresponding coefficient.* □

It follows from the above that we may choose $\tilde y(s) = \sigma^2(s)y(t(s))$.

If γ is represented as the projective image of a solution $y(t)$ of ODE (2), then the dual curve γ^* is represented as the projective image of a solution $z(t)$ of the adjoint ODE [10, p. 61], [8, p. 16]

$$[z'''(t) + 2\alpha(t)z'(t) + \alpha'(t)z(t)] - \beta(t)z(t) = 0. \tag{5}$$

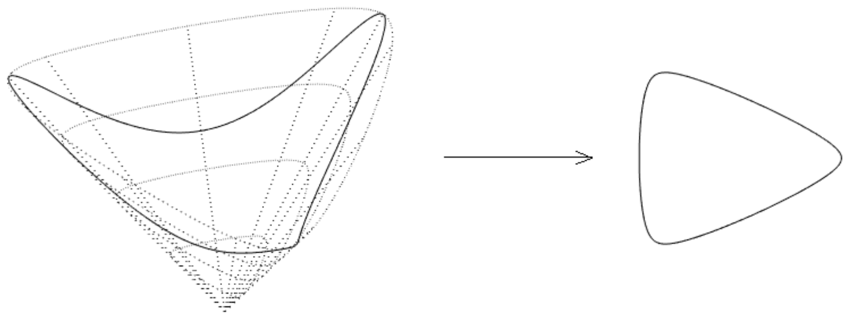

Figure 1: Solution $y(t)$ on the boundary of the cone K and its projection onto the simple closed convex curve γ in an affine chart in $\mathbb{R}P^2$.

Simple closed convex projective plane curves (i.e., without self-intersections, contained and convex in some affine chart on $\mathbb{R}P^2$) are canonically isomorphic to the manifold of boundary rays of convex proper three-dimensional cones. The solution $y(t)$ evolves on the boundary ∂K, while $z(t)$ evolves on ∂K^*, the boundary of the dual cone (see Fig. 1).

Since the curve γ is closed, we may parameterize it 2π-periodically by a variable $t \in \mathbb{R}$. In this case the coefficient functions $\alpha(t), \beta(t)$ are also 2π-periodic. The behaviour of solutions of ODEs with periodic coefficients is the subject of Floquet theory [3]. Namely, a shift of the variable t by 2π maps the solution space of ODE (3) to itself, and there exists $T \in SL(2,\mathbb{R})$ such that $x(t+2\pi) = Tx(t)$ for all $t \in \mathbb{R}$. The map T is called the *monodromy* of equation (3). The conjugacy class of the monodromy as well as the winding number of the vector-valued solution $x(t)$ of (3) around the origin over one period are invariant under reparametrizations $t \mapsto s(t)$ of γ satisfying $s(t+2\pi) = s(t) + 2\pi$, i.e., preserving the periodicity condition [8, pp. 24–25, 34–35].

Equation (3) with periodic coefficient function has been well studied and is known under the name *Hill equation*. In [6] a complete classification of the coefficient functions under the equivalence relation generated by the group of orientation-preserving diffeomorphisms of S^1 and a construction of corresponding normal forms has been achieved. The equations can be classified according to several criteria. They may be divided in stable, semi-stable and unstable ones, according to the asymptotic behaviour of the solutions, or into oscillating and non-oscillating ones, according to the behaviour of the argument of the vector-valued solution. Stable solutions are always oscillating. The normal forms of the non-oscillating and the stable equations have constant coefficient functions, while in the remaining cases their coefficient functions are sinusoidal.

In [4] it has been established that the 2π-periodic solutions of equation (4) can be seen as vector fields generating diffeomorphisms of S^1 which preserve the coefficient function in (3), and at least one non-trivial periodic solution always exists. If such a solution is nowhere zero, then it can be used to construct a diffeomorphism of S^1 which takes the coefficient function α to a constant. Moreover, this diffeomorphism is unique up to a rotation of S^1 if and only if $\alpha \neq \frac{n^2}{2}$ for all $n \in \mathbb{N}_+$. Equations with different values of the constant are non-equivalent.

Our strategy will consist in constructing diffeomorphisms of S^1 which transform the coefficient function of Hill equation (3) to a constant $\alpha \leq \frac{1}{2}$. In particular, we prove the following result.

Theorem 1.3. *Let γ be a simple closed convex projective plane curve of class C^k, $k \geq 5$, without inflection points. Then there exists a 2π-periodic parametrization of γ of class C^{k-1} by a real variable t and a 2π-periodic lift $y : \mathbb{R} \to \mathbb{R}^3$ of γ of class C^{k-2} such that $y(t)$ is a solution of ODE (2) with $\alpha \equiv const$. Here the value of the constant α is uniquely determined by the curve γ.*

Since the classification results in [6, 4] have been established in the C^∞ setting, we shall provide an independent proof.

We shall now briefly summarize the contents of the paper. First we explicitly describe the solution $z(t)$ of the adjoint ODE (5) in terms of $y(t)$ (Lemma 2.1). Next we show that during each period of length 2π the projective image of the vector-valued solution $w(t)$ of ODE (4) can make at most one turn around the ellipse ς on which it evolves. Equivalently, the solution $x(t)$ of ODE (3) can make at most one half of a turn around the origin (Lemma 2.2). This heavily restricts the behaviour of the solution $x(t)$ (Lemma 2.3) and allows to construct a reparametrization of γ which makes the coefficient α constant (Theorem 1.3). The value of the constant α depends on the eigenvalues of the monodromy T of ODE (3) and is hence uniquely determined by the curve γ. It follows in particular that in general the Forsyth-Laguerre parametrization cannot be extended to the whole closed curve γ (Corollary 2.5).

We call a 2π-periodic parametrization of γ *balanced* if the corresponding coefficient function α in (2) is constant.

2 Balanced parametrizations

Let γ be a simple closed convex projective plane curve of class C^k, $k \geq 5$, without inflection points. Let the lift $y(t)$ of γ be a 2π-periodic vector-valued solution of ODE (2) such that $\det(y'', y', y) \equiv 1$. The 2π-periodic coefficient functions α, β are then of class C^{k-4}, C^{k-5}, respectively, and y is of class C^{k-2}.

Denote $Y = (y'' + \alpha y, y', y) \in SL(3, \mathbb{R})$, then (2) is equivalent to the matrix-valued ODE

$$Y' = Y \cdot A_-, \tag{6}$$

where for convenience we denoted $A_\pm = \begin{pmatrix} 0 & 1 & 0 \\ -\alpha & 0 & 1 \\ \pm\beta & -\alpha & 0 \end{pmatrix}$. We now describe the dual objects in terms of the matrix Y.

Lemma 2.1. *Assume above conditions. Let γ^* the dual projective curve of γ. There exists a vector-valued solution z of (5) which is a lift of γ^* and satisfies $\det(z'', z', z) \equiv 1$. The matrix $Z = (z'' + \alpha z, z', z) \in SL(3, \mathbb{R})$ is given by $Z = Y^{-T}Q$ with*

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof. Denote the matrix product $Y^{-T}Q$ by Z and let z be its third column. Clearly Z is unimodular and 2π-periodic. In particular, z is non-zero everywhere. Further, by (6) the product Z satisfies the differential equation

$$Z' = -Y^{-T}(YA_-)^T Y^{-T} Q = -ZQ^{-1}A_-^T Q = Z \cdot A_+.$$

It follows that $Z = (z'' + \alpha z, z', z)$ and z is a solution of ODE (5). It follows also that $\det(z'', z', z) \equiv 1$. Finally, we have $Y^T Z = Q$, which implies $\langle y(t), z(t) \rangle = \langle y'(t), z(t) \rangle = 0$ for all t. Hence the vector $z(t)$ is orthogonal to the plane spanned by $y(t)$ and $y'(t)$, and the projective image of $z(t)$ is the corresponding point $\gamma^*(t)$ on the dual projective curve. Thus z satisfies all required conditions. $\square$

Note that the dual curve γ^* is also simple closed convex and of class C^k without inflection points. Let now $t_0 \in \mathbb{R}$ and set $y_0 = y(t_0)$, $z_0 = z(t_0)$. Define the scalar C^{k-2} functions $\mu(t) = \langle y(t), z_0 \rangle$, $\nu(t) = \langle y_0, z(t) \rangle$. By convex duality these functions are nonnegative, and $\mu(t) = 0$ or $\nu(t) = 0$ if and only if $t - t_0$ is an integer multiple of the period 2π.

Assume the notations of Lemma 2.1. We have $ZQY^T = I$ and hence

$$\begin{aligned} 0 = \langle y_0, z_0 \rangle = y_0^T ZQY^T z_0 &= (\nu'' + \alpha\nu, \nu', \nu)Q(\mu'' + \alpha\mu, \mu', \mu)^T \\ &= \nu\mu'' + 2\alpha\nu\mu + \mu\nu'' - \nu'\mu'. \end{aligned}$$

For $t_0 < t < t_0 + 2\pi$ define the C^{k-3} functions $\xi = \frac{\mu'}{\mu}$, $\theta = \frac{\nu'}{\nu}$. Dividing the above relation by $\mu\nu$ and expressing the result in terms of ξ, θ we obtain

$$\xi' + \theta' + \xi^2 - \xi\theta + \theta^2 + 2\alpha = 0.$$

Introducing the variable $\psi = \frac{1}{4}(\xi + \theta)$ and taking into account $\xi^2 - \xi\theta + \theta^2 = 4\psi^2 + \frac{3}{4}(\xi - \theta)^2$ we obtain the differential inequality

$$\psi' + \psi^2 + \frac{\alpha}{2} = -\frac{3}{16}(\xi - \theta)^2 \leq 0. \tag{7}$$

Lemma 2.2. *Assume the conditions at the beginning of this section. Let $t_0 \in \mathbb{R}$ be arbitrary, and let $x(t)$ be a non-trivial scalar solution of ODE (3). Then $x(t)$ cannot have two distinct zeros in the interval $(t_0, t_0 + 2\pi)$. If $x(t_0) = x(t_0 + 2\pi) = 0$, then $\beta \equiv 0$ and γ is an ellipse.*

The first assertion follows by virtue of [2, Proposition 9, p. 130] from the existence of a function $\psi(t)$ satisfying (7) on $(t_0, t_0 + 2\pi)$. We shall, however, give an elementary proof below.

Proof. Let $t_m \in (t_0, t_0 + 2\pi)$ be arbitrary and define the positive function

$$q(t) = \exp\left(\int_{t_m}^{t} \psi(t)\, dt\right) = \left(\frac{\mu(t)\nu(t)}{\mu(t_m)\nu(t_m)}\right)^{1/4}$$

on $(t_0, t_0 + 2\pi)$, where $\psi(t)$ is the function from (7). Then we obtain $q'' + \frac{\alpha}{2}q = (\psi' + \psi^2 + \frac{\alpha}{2})q \leq 0$.

Let $x(t)$ be an arbitrary non-trivial solution of ODE (3) on $(t_0, t_0 + 2\pi)$ and consider the function $r(t) = x'(t)q(t) - x(t)q'(t)$. We have $r' = x''q - xq'' = -x(q'' + \frac{\alpha}{2}q)$.

Suppose for the sake of contradiction that $x(t_1) = x(t_2) = 0$ for $t_0 < t_1 < t_2 < t_0 + 2\pi$ and $x(t) > 0$ for all $t \in (t_1, t_2)$. Then $x'(t_1) > 0$, $x'(t_2) < 0$, and hence $r(t_1) > 0$, $r(t_2) < 0$. But $r'(t) \geq 0$ on (t_1, t_2), a contradiction. The case when $x(t)$ is negative on (t_1, t_2) is treated similarly. This proves the first claim.

Since t_0 is arbitrary, it follows that no non-trivial solution of ODE (3) can have two consecutive zeros at a distance strictly smaller than 2π.

Let now $x(t)$ be a non-trivial solution of ODE (3) such that $x(t_0) = x(t_0 + 2\pi) = 0$. Then $x(t)$ has constant sign on $(t_0, t_0 + 2\pi)$, and $r'(t)$ is either nonnegative or non-positive, depending on the sign of x. In any case the function $r(t)$ is monotonous on $(t_0, t_0 + 2\pi)$. Note that $q(t)$ and $q'(t)$ can be continuously prolonged to t_0 and $t_0 + 2\pi$ and the limits of $q(t)$ vanish. We hence have $\lim_{t\to t_0} r(t) = \lim_{t \to t_0+2\pi} r(t) = 0$. It follows that $r \equiv 0$, $r' \equiv 0$, and therefore $q'' + \frac{\alpha}{2}q \equiv 0$ on $(t_0, t_0 + 2\pi)$. But then inequality (7) is actually an equality and $\xi \equiv \theta$. Then there exists a constant $c > 0$ such that $\mu \equiv c\nu$. But $\mu(t)$ is a solution of ODE (2), while $\nu(t)$ and hence also $c\nu(t)$ is a solution of (5). Subtracting (5) from (2) with z, y replaced by μ, respectively, we obtain $2\beta(t)\mu(t) = 0$ on $(t_0, t_0 + 2\pi)$. It follows that $\beta \equiv 0$, $y(t)$ is a solution of ODE (4) and hence γ is an ellipse. This completes the proof. $\square$

Lemma 2.2 allows to restrict the global behaviour of the solutions of ODE (3).

Lemma 2.3. *Assume the conditions at the beginning of this section. Then exactly one of the following cases holds:*

(i) There exists a solution $x(t)$ of ODE (3), normalized such that $\det(x, x') \equiv 1$, that is contained in the open positive orthant and crosses each ray of this orthant exactly once, and whose monodromy equals $T = \mathrm{diag}(\lambda^{-1}, \lambda)$ for some $\lambda > 1$.

(ii) There exists a solution $x(t)$ of ODE (3), normalized such that $\det(x, x') \equiv 1$, that is contained in the open right half-plane and crosses each ray of this half-plane exactly once, and whose monodromy equals $T = \begin{pmatrix} 1 & 0 \\ 2\pi & 1 \end{pmatrix}$.

(iii) There exists a solution $x(t)$ of ODE (3), normalized such that $\det(x, x') \equiv 1$, that is bounded and turns infinitely many times around the origin, and whose monodromy equals $T = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ for some $\varphi \in (0, \pi)$. For every $t_0 \in \mathbb{R}$ the solution turns by an angle of φ around the origin in the interval $[t_0, t_0 + 2\pi]$.

(iv) There exists a 4π-periodic solution $x(t)$ of ODE (3), normalized such that $\det(x, x') \equiv 1$, and whose monodromy equals $T = -I$.

The curve γ is an ellipse if and only if case (iv) holds.

Proof. Let $x(t)$ be an arbitrary solution of ODE (3) with linearly independent components, normalized such that $\det(x, x') \equiv 1$. Any other such solution can then be obtained by the action of an element of $SL(2, \mathbb{R})$. The solution x turns counter-clockwise around the origin and intersects every ray transversally.

First we shall treat the case when γ is not an ellipse. By Lemma 2.2 every scalar solution of ODE (3) has its consecutive zeros placed at distances strictly larger than 2π. Hence x turns by an angle strictly less than π in any time interval of length 2π. In particular, it follows that the solution $x(t)$ cannot cross any 1-dimensional eigenspace of the monodromy T. Indeed, suppose that for some $t_0 \in \mathbb{R}$ the vector $x(t_0)$ is an eigenvector of T. Then $x(t_0 + 2\pi) = Tx(t_0)$ is a positive or negative multiple of $x(t_0)$, and x must have made at least half of a turn around the origin in the interval $[t_0, t_0 + 2\pi]$, a contradiction.

We shall now distinguish several cases according to the spectrum of the monodromy T of ODE (3). Let $T \in SL(2,\mathbb{R})$ be such that $x(t + 2\pi) = Tx(t)$ for all T. If $\tilde{x} = Ax$ for some $A \in SL(2,\mathbb{R})$, then $\tilde{x}(t + 2\pi) = \tilde{T}\tilde{x}(t)$ with $\tilde{T} = ATA^{-1}$. We may hence conjugate T with an arbitrary unimodular matrix by switching to another solution x.

Case 1: The eigenvalues of T are given by λ, λ^{-1} for some $\lambda > 1$. By conjugation with a unimodular matrix we may achieve $T = \mathrm{diag}(\lambda^{-1}, \lambda)$. Since $x(t)$ cannot cross the axes, it must be confined to an open quadrant. For every point q in the second or fourth open quadrant the vector Tq has a polar angle strictly less than that of q. But $x(t)$ turns in the counter-clockwise direction, and hence cannot be contained in these quadrants. By possibly multiplying x by -1 we may hence achieve that x is contained in the open positive orthant. Now for any $t_0 \in \mathbb{R}$ the angles of the vectors $T^k x(t_0)$ tend to $\frac{\pi}{2}$ and those of $T^{-k}x(t_0)$ to 0 as $k \to +\infty$. Therefore the angles of $x(t)$ sweep the interval $(0, \frac{\pi}{2})$ as t sweeps the real line. This is the situation described in case (i) of the lemma.

Case 2: The eigenvalues of T equal 1. Since $x(t)$ cannot be an eigenvector of T for any t, we must have $T \neq I$ and the Jordan normal form of T contains a single Jordan cell. By conjugation with a unimodular matrix we may then achieve that $T = \begin{pmatrix} 1 & 0 \\ \pm 2\pi & 1 \end{pmatrix}$. Since $x(t)$ cannot cross the vertical axis, it must be contained in the left or right open half-plane. By multiplying by -1 we may assume the solution is contained in the right half-plane. Now if the $(2,1)$ element in T equals -2π, then for every point q in the open right half-plane the vector Tq has a polar angle strictly less than that of q. This is in contradiction with the counter-clockwise movement of x, and this case cannot appear. Hence the $(2,1)$ element in T equals 2π. Then for any $t_0 \in \mathbb{R}$ the angles of the vectors $T^k x(t_0)$ tend to $\frac{\pi}{2}$ and those of $T^{-k}x(t_0)$ to $-\frac{\pi}{2}$ as $k \to +\infty$. Therefore the angles of $x(t)$ sweep the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ as t sweeps the real line. This is the situation described in case (ii) of the lemma.

Case 3: The eigenvalues of T equal $e^{\pm i\varphi}$ for $\varphi \in (0, \pi)$. By conjugation with an element in $SL(2,\mathbb{R})$ we may achieve that $T = \begin{pmatrix} \cos\varphi & \mp\sin\varphi \\ \pm\sin\varphi & \cos\varphi \end{pmatrix}$. If the $(2,1)$ element of T has negative sign, then for every $q \neq 0$ the angle of Tq equals $2\pi - \varphi$ plus the angle of q. Since x moves counter-clockwise, it must hence sweep an angle of at least $2\pi - \varphi > \pi$ on any interval of length 2π, which is not possible. Hence the $(2,1)$ element of T has positive sign, and for every $q \neq 0$ the angle of Tq equals φ plus the angle of q. Since x cannot make a complete turn around the origin in an interval of length 2π, the angle swept by the solution on any such interval equals φ. Finally note that since T acts by a rotation, the norm of the solution x is 2π-periodic and hence uniformly bounded. This is the situation described in case (iii) of the lemma.

Case 4: The eigenvalues of T equal -1. Similarly to Case 2 we have $T \neq -I$, and the Jordan normal form of T consists of a single Jordan cell. The eigenspace to the eigenvalue -1 then divides $\mathbb{R}^2$ in two half-planes. For every q in one of the open half-planes, the point Tq lies in the other open half-plane. Hence the solution $x(t)$ must cross the eigenspace, leading to a contradiction. Hence this case does not occur.

Case 5: The eigenvalues of T equal $-\lambda, -\lambda^{-1}$ for some $\lambda > 1$. By conjugation with a unimodular matrix we may achieve $T = \mathrm{diag}(-\lambda^{-1}, -\lambda)$. Similarly to Case 1 the solution $x(t)$ must then be contained in some open quadrant. But the map T maps every quadrant to the opposite quadrant. Hence x must cross the axes, which leads to a contradiction. Thus this case does not occur either.

We now consider the case when γ is an ellipse. By Lemma 2.2 we have $\beta \equiv 0$ and (2), (4) represent the same ODE. Since all solutions y of ODE (2) are 2π-periodic, the solutions w of (4) are also 2π-periodic. But the solutions w are homogeneous quadratic functions of the solutions x of ODE (3). Hence the latter are 4π-periodic, and $T^2 = I$. If $T = I$, then every two consecutive zeros of every non-trivial scalar solution of ODE (3) have a distance strictly smaller than 2π, leading to a contradiction with Lemma 2.2. Hence $T = -I$, and we are in the situation described in case (iv) of the lemma.

This completes the proof. □

Remark 2.4. The cases i) — iv) in the formulation of the lemma correspond to the unstable non-oscillating, semi-stable non-oscillating, stable with $\Theta < 1$, and stable with $\Theta = 1$ cases, correspondingly, in the classification in [6]. The cases 4 and 5 in the proof correspond to the semi-stable and unstable oscillating cases in [6].

Corollary 2.5. *Assume the conditions at the beginning of this section. If the eigenvalues of the monodromy of ODE* (3) *differ from 1, then the curve γ does not possess a global periodic Forsyth-Laguerre parametrization.*

Proof. Suppose γ possesses a periodic Forsyth-Laguerre parametrization by a variable s. In this parametrization any non-zero vector-valued solution $\tilde{x}(s)$ of ODE (3) with independent components is a straight affine line, and hence sweeps a total angle of π in the plane.

Let now γ be parameterized 2π-periodically by a variable t. Every non-zero vector-valued solution $x(t)$ of ODE (3) with independent components must also sweep a total angle of π. From Lemma 2.3 it follows that the monodromy of ODE (3) has eigenvalues equal to 1. $\square$

We are now in a position to construct the reparametrization $t \mapsto s(t)$ which makes the coefficient α in ODE (2) constant.

of Theorem 1.3. We shall begin with an arbitrary regular 2π-periodic parametrization of γ of class C^k. As laid out in Section 1, there exists a 2π-periodic lift $y(t)$ of γ which solves ODE (2) with some 2π-periodic functions $\alpha(t)$, $\beta(t)$ of class C^{k-4}, C^{k-5}, respectively. The coefficient function α gives rise to ODE (3). We shall construct a 2π-periodic parametrization of γ by a new variable s from the vector-valued C^{k-2} solutions $x(t) = (x_1(t), x_2(t))$ of ODE (3) described in Lemma 2.3. Note that if we write $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, then the condition $\det(x, x') \equiv 1$ implies $r^2\phi' \equiv 1$ and $\phi' = r^{-1/2}$. Since $r(t)$ is of class C^{k-2}, the angle ϕ is of class C^{k-1}. We consider the four cases (i) — (iv) in Lemma 2.3 separately.

Case (i): Set $s(t) = \frac{\pi}{\log\lambda}\log\frac{x_2(t)}{x_1(t)}$. Note that s is an analytic function of the angle ϕ and hence $s(t)$ is a C^{k-1} function. We have $s(t+2\pi) = \frac{\pi}{\log\lambda}\log\frac{\lambda x_2(t)}{\lambda^{-1}x_1(t)} = s(t) + 2\pi$, and the new parameter s parameterizes γ 2π-periodically. Set further $c = \frac{\pi}{\log\lambda} > 0$ and $\tilde{\alpha} = -\frac{1}{2c^2} < 0$. Then the vector-valued function $\tilde{x}(s) = (\sqrt{c}\lambda^{-s/2\pi}, \sqrt{c}\lambda^{s/2\pi})$ obeys the differential equation $\frac{d^2\tilde{x}}{ds^2} + \frac{\tilde{\alpha}}{2}\tilde{x} = 0$ and we have $\frac{\tilde{x}_2(s(t))}{\tilde{x}_1(s(t))} = \lambda^{s(t)/\pi} = \frac{x_2(t)}{x_1(t)}$ for all t. Moreover, $\det(\tilde{x}, \frac{d\tilde{x}}{ds}) \equiv 1$. By Corollary 1.2 the coefficient α in ODE (2) in the new coordinate s identically equals the constant $\tilde{\alpha}$. The coefficient β in the new variable is given by $\tilde{\beta}(s) = \beta(t)(\frac{ds}{dt})^{-3}$, because β transforms as the coefficient of a cubic differential. Hence $\tilde{\beta}(s)$ is as $\beta(t)$ a C^{k-5} function. Therefore the solution $\tilde{y}(s)$ of ODE (2) in the variable s is of class C^{k-2}.

Case (ii): Set $s(t) = \frac{x(t_2)}{x(t_1)}$. Again s is an analytic function of the angle ϕ and $s(t)$ is a C^{k-1} function. We have $s(t+2\pi) = \frac{2\pi x(t_1) + x(t_2)}{x(t_1)} = s(t) + 2\pi$, and s parameterizes γ 2π-periodically. Define $\tilde{x}(s) = (1, s)$, then $\det(\tilde{x}, \frac{d\tilde{x}}{ds}) \equiv 1$, $\frac{d^2\tilde{x}}{ds^2} = 0$, and $\frac{\tilde{x}_2(s)}{\tilde{x}_1(s)} = \frac{x(t_2)}{x(t_1)}$. By Corollary 1.2 the coefficient α in ODE (2) in the new coordinate s identically equals zero. As in the previous case the coefficient $\tilde{\beta}(s)$ is a C^{k-5} function and the solution $\tilde{y}(s)$ of ODE (2) in the variable s is of class C^{k-2}.

Case (iii): Set $s(t) = \frac{2\pi}{\varphi}\phi(t)$. Again s is a C^{k-1} function and $s(t+2\pi) = \frac{2\pi}{\varphi}(\phi(t) + \varphi) = s(t) + 2\pi$, and s parameterizes γ 2π-periodically. Define $c = \frac{2\pi}{\varphi}$, $\tilde{\alpha} = \frac{2}{c^2}$, and $\tilde{x}(s) = (\sqrt{c}\cos\frac{s}{c}, \sqrt{c}\sin\frac{s}{c})$. Then $\det(\tilde{x}, \frac{d\tilde{x}}{ds}) \equiv 1$, $\frac{d^2\tilde{x}}{ds^2} + \frac{\tilde{\alpha}}{2}\tilde{x} = 0$, and the angles of $x(t)$ and $\tilde{x}(s)$ both equal ϕ. By Corollary 1.2 the coefficient α in ODE (2) in the new coordinate s identically equals the constant $\tilde{\alpha}$. As in the previous case the coefficient $\tilde{\beta}(s)$ is a C^{k-5} function and the solution $\tilde{y}(s)$ of ODE (2) in the variable s is of class C^{k-2}.

Case (iv): The curve γ is an ellipse, and by an appropriate choice of the coordinate basis in $\mathbb{R}^3$ we may achieve that γ is the projective image of the vector-valued function $y(t) = (1, \cos t, \sin t)$. This function is a solution of ODE (2) with $\alpha \equiv \frac{1}{2}$, $\beta \equiv 0$, and the variable t parameterizes γ analytically and 2π-periodically.

Finally we show that the value of the constant α is uniquely determined by γ. Let the lift $y(t)$ of γ be a 2π-periodic solution of ODE (2) with constant coefficient α. Let $x(t)$ be the solution from Lemma 2.3.

If $\alpha < 0$, then $x(t)$ must be a hyperbola, hence case (i) is realized, and α relates to the spectrum of the monodromy T of ODE (3) by $\alpha = -\frac{\log^2\lambda}{2\pi^2}$.

If $\alpha = 0$, then by Corollary 2.5 the eigenvalues of T equal 1.

If $\alpha \in (0, \frac{1}{2})$, then $x(t)$ must be an ellipse and sweeps an angle strictly less than π in any interval of length 2π. Hence case (iii) is realized, and α is related to the spectrum of T by $\alpha = \frac{\varphi^2}{2\pi^2}$.

If $\alpha \geq \frac{1}{2}$, then $x(t)$ must also be an ellipse and sweeps an angle of at least π in any interval of length 2π. Hence case (iv) is realized, $x(t)$ sweeps an angle of exactly π, and $\alpha = \frac{1}{2}$.

In any case α is uniquely determined by the spectrum of T. However, the spectrum of T depends only on γ. Hence α is also uniquely determined by γ. $\square$

Definition 2.6. Let γ be a simple closed convex projective plane curve of class C^k, $k \geq 5$, without inflection points. We call a 2π-periodic parametrization of γ by a real variable t *balanced* if there exists a 2π-periodic lift $y(t)$ of γ to $\mathbb{R}^3$ which is a vector-valued solution of ODE (2) with $\alpha \equiv$ *const*.

By Theorem 1.3 a balanced parametrization always exists. In the case of non-quadratic curves the balanced parametrization is unique up to a shift of the variable t by [4, Lemma 2], and hence defines an invariant metric on the curve. For an ellipse every two balanced parametrizations are related by a projective transformation.

Acknowledgements

The author would like to thank the referee for a thorough review and for pointing out relevant literature on normal forms of ODEs.

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